Free Vibration of an Un-damped Torsional System

let

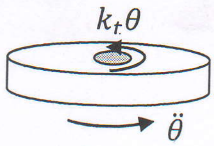
$k_t = \frac{GI_p}{l}$:– torsional stiffness of shaft

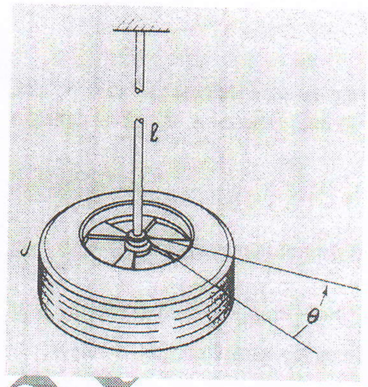
where

G:– shear modulus of rigidity N/m^2

l:– length (m)

d:– diameter (m)





I_p:–polar moment of inertia of the circular cross-sectional area

$I_p = \frac{\pi d^4}{32}$ (m^4)

Applying Newton's second law $\sum M_C = J\ddot{\theta}$ J:– mass moment of inertia

$-k_t\theta = J\ddot{\theta}$ or $J\ddot{\theta} + k_t\theta = 0$

$\ddot{\theta} + \frac{k_t}{J}\theta = 0$ **(Equation of Motion)**

$\omega_n = \sqrt{\frac{k_t}{J}}$ **(Natural Frequency)**

Solution of Equation of Motion

Equation of motion

$\ddot{x} + \frac{k}{m}x = 0$ $\ddot{x} + \omega_n^2 x = 0$ **(Equation of Motion for Translation Motion)**

$\ddot{\theta} + \frac{k_t}{J}\theta = 0$ $\ddot{\theta} + \omega_n^2\theta = 0$ **(Equation of Motion for Rotational Motion)**

is second order differential equation which has the solution in the following forms

Form 1

$$x(t) = C\cos(\omega_n t - \phi) \qquad \cdots\cdots\cdots (2)$$

where C and ϕ are constants

C:- amplitude

ϕ:- phase angle

to find the constants we need the following initial conditions

$x(0) = x_0$:- initial displacement and $\dot{x}(0) = v_0$:- initial velocity

from Equation (2), let $t = 0$ $\Rightarrow$ $x(t = 0) = x_0$

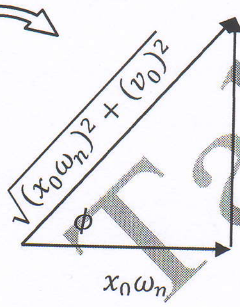
$$x_0 = C\cos(\phi) \qquad\qquad C = \frac{x_0}{\cos(\phi)}$$

let $t = 0$ $\quad \dot{x}(t=0) = v_0$

from Equation (1) $\quad \dot{x}(t) = -C\ \omega_n \sin(\omega_n t - \phi) \qquad v_0 = C\ \omega_n \sin(\phi)$

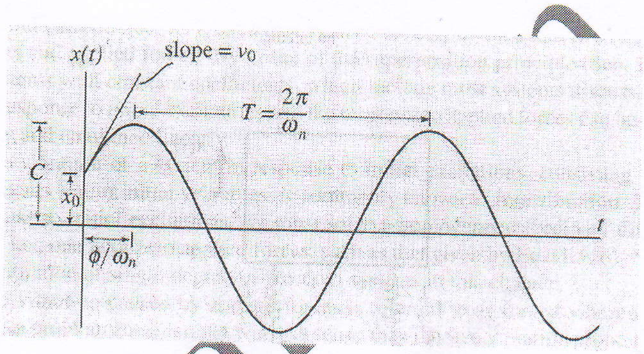
or $\quad v_0 = \frac{x_0}{\cos(\phi)}\omega_n \sin(\phi) = x_0\omega_n\ \tan\phi$

$$\tan\phi = \frac{v_0}{x_0\omega_n} \qquad \phi = \tan^{-1}\frac{v_0}{x_0\omega_n}$$



$$C = \frac{x_0}{\cos(\phi)} = \frac{x_0}{\frac{x_0\omega_n}{\sqrt{(\omega_n x_0)^2+(v_0)^2}}} =$$

or $\quad C = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}$



Form 2

$$x(t) = C_1\cos\omega_n t + C_2\sin\omega_n t \qquad \cdots\cdots\cdots (3)$$

from Equation (3), let $t = 0$ $\qquad x(t=0) = x_0$

$x_0 = C_1$

$\dot{x}(t) = -C_1\omega_n\sin\omega_n t + C_2\omega_n\cos\omega_n t$

let $t = 0$ $\quad \dot{x}(t=0) = v_0$

$v_0 = C_2\omega_n \qquad C_2 = \frac{v_0}{\omega_n}$

$$x(t) = x_0\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

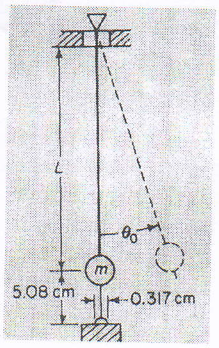
T:- The period of oscillation $(sec) = \frac{2\pi}{\omega_n}$

Note :- the natural frequency can also be defined as the reciprocal of the period

$$f_n = \frac{1}{T}\ \text{Hz}$$

Example

A chronograph is to be operated by a 2- second pendulum of length L. A platinum wire attached to the bob completes the electric timing circuit through a drop of mercury as it swing through the lowest point. (a)- What should be the length L of the pendulum, (b)-If the platinum wire is in contact with the mercury for 0.3175 cm of the swing what must be the amplitude θ to limit the duration of contact to 0.01 second? (Assume that the velocity during contact is constant and that the amplitude of oscillation is small).

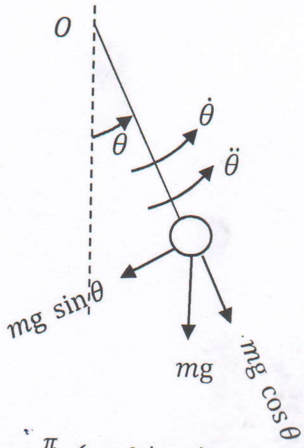


Solution

$\sum M_o = J_o\ddot{\theta}$ where $J_o = mL^2$

$$-mg\sin\theta \cdot L = mL^2\ddot{\theta}$$

for small angle $\sin\theta = \theta$



$$L\ddot{\theta} + g\,\theta = 0$$

then the equation of motion is $\ddot{\theta} + \frac{g}{L}\theta = 0$

where $\omega_n = \sqrt{\frac{g}{L}}$ ……… (a)

but $T = 2 \times 2 = 4\ sec$ = The period of oscillation $= \frac{2\pi}{\omega_n}$ $\Rightarrow$ $\omega_n = \frac{\pi}{2}\ (rad/sec)$

the initial velocity is $v = \frac{0.315\times10^{-2}}{0.01} = 0.315\ m/s$

from Equation (a) $\frac{\pi}{2} = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.81}{L}}$ $\Rightarrow$ $L = 3.976\ m$

$\dot{\theta}_0$ = initial angular velocity$= \frac{v}{r} = \frac{0.315}{3.976+0.0508} = 0.07885\ rad/sec$

For rotational motion the Equation of motion becomes to

$$\theta(t) = C\cos(\omega_n t - \phi) \qquad \text{………} (b)$$

$$C = \sqrt{\theta_0^2 + \left(\frac{\dot{\theta}_0}{\omega_n}\right)^2} \qquad \phi = \tan^{-1}\frac{\dot{\theta}_0}{\theta_0\omega_n}$$

when $\dot{\theta} = \dot{\theta}_0 = 0.07885\ rad/sec$ $\quad \theta_0 = 0$

$$C = \sqrt{0 + \left(\frac{0.07885}{\pi/2}\right)^2} = 0.0502\ rad \qquad \phi = \frac{\pi}{2} = 90^\circ$$

then Equation (b) becomes to

$$\theta(t) = 0.0502 \sin\left(\frac{\pi}{2}t\right)$$ at maximum amplitude $t = 1$

$$\theta_{max}(t) = 0.0502$$

Viscously Damped Single Degree of Freedom

A damping is assumed to have neither mass nor elasticity and damping force exists only if there is relative velocity between the two ends of the damper. The energy or work input to a damper is converted into heat or sound.

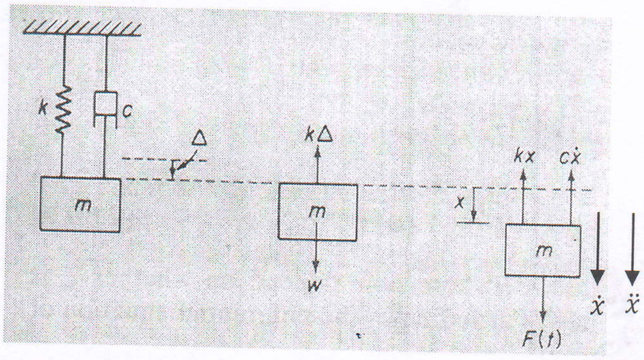
F_d :- damping force

$F_d = c\dot{x}$ where c :- **damping constant** or **coefficient of viscous damping**

From Newton's second law

$\sum \mathbf{F} = m\ddot{x}$ where $\ddot{x}$ acceleration of the mass

Then, **From Free Body Diagram**



$$-kx - c\dot{x} = m\ddot{x}$$

Note that $k\Delta = w$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \quad \cdots\cdots\cdots (1)$$

$$\frac{k}{m} = \omega_n^2$$

$$\frac{c}{m} = \frac{c}{m} \times \frac{2\omega_n}{2\omega_n} = \underbrace{\frac{c}{2\omega_n m}}_{\zeta} 2\omega_n = 2\zeta\omega_n$$

where $\boxed{\zeta = \frac{c}{2\omega_n m}}$ = damping ratio [dimensionless]

Note that damping ratio ζ can be written as $\zeta = \frac{c}{2m\sqrt{\frac{k}{m}}} = \boxed{\frac{c}{2\sqrt{km}}}$

then Equation (1) can be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad \cdots\cdots\cdots (2)$$

Equation (2) is homogeneous 2-nd order D.E

Let the solution $x(t) = e^{\lambda t}$ $\dot{x}(t) = \lambda e^{\lambda t}$ $\ddot{x}(t) = \lambda^2 e^{\lambda t}$

Substituting these into Equation (2)

$$e^{\lambda t}(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2) = 0 \qquad e^{\lambda t} \neq 0 \quad \text{then}$$

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \qquad \cdots\cdots\cdots (3) \quad \text{[characteristic equation]}$$

The two roots of (3) are

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\ \ \omega_n \qquad \cdots\cdots\cdots (*)$$

when $= 0$, the roots λ_1 and λ_2 correspond to the points $i\omega_n$ and $-i\omega_n$, in this the motion represents harmonic motion with natural frequency ω_n. This case was discussed in p (21).

1- **Underdamped system** $0 < \zeta < 1$

In this case the two roots of Equation $(*)$ are

$$\lambda_{1,2} = -\zeta\omega_n \pm i\sqrt{1 - \zeta^2}\ \ \omega_n = -\zeta\omega_n \pm i\,\omega_d$$

where $\omega_d = \sqrt{1 - \zeta^2}\ \ \omega_n$ = damped natural frequency

Note that $\omega_d < \omega_n$ and $T_d > T$ since $T = \frac{2\pi}{\omega_n}$ and $T_d = \frac{2\pi}{\omega_d}$

Then the general solution of equation (2) becomes to

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

or $x(t) = C_1 e^{(-\zeta\omega_n + i\,\omega_d)t} + C_2 e^{(-\zeta\omega_n - i\,\omega_d)t}$ since $[e^{i\,\omega_d t} = \cos\omega_d t + i\sin\omega_d t]$

$$\boxed{x(t) = e^{-\zeta\omega_n t}[C_1\cos\omega_d t + C_2\sin\omega_d t]} \qquad \textbf{[Form 1]}$$

where C_1 and C_2 are two constant where can determined from initial conditions.

$$x(t = 0) = x_0 \qquad \text{and} \qquad \dot{x}(t = 0) = \dot{x}_0 = v_0$$

from first condition $x_0 = C_1$

$$x(t) = e^{-\zeta\omega_n t}[x_0\cos\omega_d t + C_2\sin\omega_d t] \qquad \cdots\cdots\cdots (4)$$

to find C_2 differentiate Equation (4) with respect to time t

$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}[x_0\cos\omega_d t + C_2\sin\omega_d t] + e^{-\zeta\omega_n t}[-x_0\omega_d\sin\omega_d t + C_2\omega_d\cos\omega_d t]$$

then from second condition $\dot{x}(t = 0) = \dot{x}_0 = v_0$

$$v_0 = -\zeta\omega_n x_0 + C_2\omega_d \qquad C_2 = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$$

Equation (4) becomes to

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{v_0+\zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right]$$

$$\boxed{x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)}$$ **[Form 2]**

From the first condition $x(t=0) = x_0$

$$x_0 = C\cos(\phi) \qquad C = \frac{x_0}{\cos(\phi)}$$

$$\dot{x}(t) = -\,Ce^{-\zeta\omega_n t}\,\omega_d\sin(\omega_n t - \phi) - C\zeta\omega_n e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

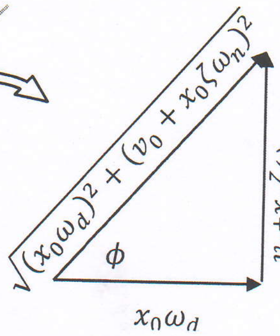
let $t = 0$ $\quad \dot{x}(t=0) = v_0$

$$v_0 = -C\omega_d\sin(-\phi) - C\zeta\omega_n\cos(-\phi)$$

$$v_0 = C\omega_d\sin(\phi) - C\zeta\omega_n\cos(\phi)$$

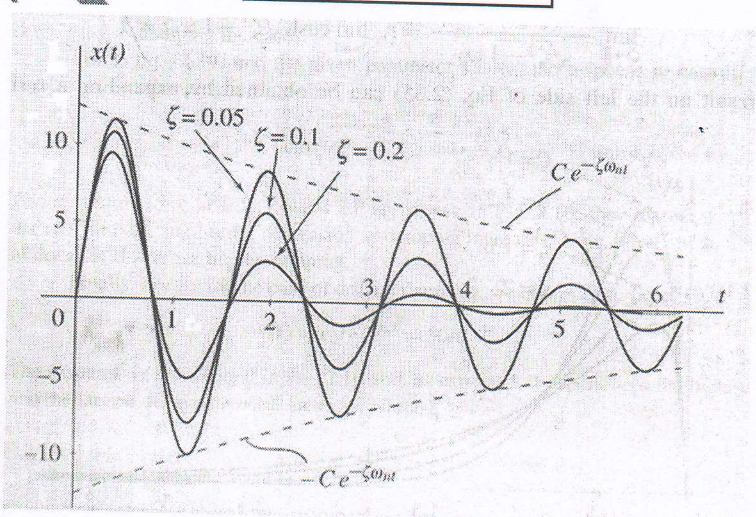
$$v_0 = \frac{x_0}{\cos(\phi)}\omega_d\sin(\phi) - x_0\zeta\omega_n \quad\Rightarrow\quad \tan\phi = \frac{v_0+x_0\zeta\omega_n}{x_0\omega_d}$$

$$\boxed{\phi = \tan^{-1}\frac{v_0+x_0\zeta\omega_n}{x_0\omega_d}}$$



$$C = \frac{x_0}{\cos(\phi)} = \frac{x_0}{\dfrac{x_0\omega_d}{\sqrt{(x_0\omega_d)^2+(v_0+x_0\zeta\omega_n)^2}}}$$

or $$C = \frac{\sqrt{(x_0\omega_d)^2+(v_0+x_0\zeta\omega_n)^2}}{\omega_d} = \boxed{\sqrt{(x_0)^2+\left(\frac{v_0+x_0\zeta\omega_n}{\omega_d}\right)^2}}$$



2- **Critical system** $\zeta = 1$

In this case the two roots of Equation (*) are

$$\lambda_{1,2} = \lambda = -\omega_n$$

Then the general solution of equation (2) becomes to

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \qquad \cdots\cdots\cdots (5)$$

where C_1 and C_2 are two constant where can determined from initial conditions

$$x(t=0) = x_0 \qquad \text{and} \qquad \dot{x}(t=0) = \dot{x}_0 = v_0$$

from first condition $x_0 = C_1$

$$x(t) = (x_0 + C_2 t)e^{-\omega_n t}$$

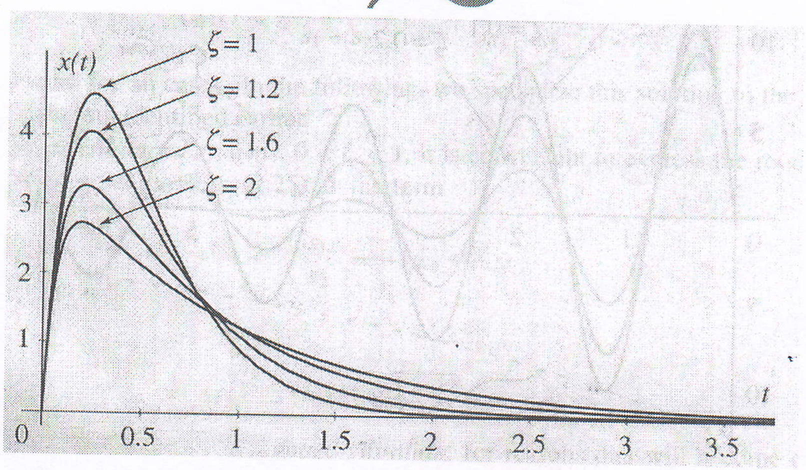
to find C_2 differentiate Equation (5) with respect to time t

$$\dot{x}(t) = -\omega_n x_0 e^{-\omega_n t} + C_2 e^{-\omega_n t} - \omega_n C_2 t e^{-\omega_n t}$$

then from second condition $\dot{x}(t=0) = \dot{x}_0 = v_0$

$$v_0 = -\omega_n x_0 + C_2 \qquad \Rightarrow \qquad C_2 = v_0 + \omega_n x_0$$

$$\boxed{x(t) = [x_0 + (\omega_n x_0 + v_0)t]\, e^{-\omega_n t}}$$



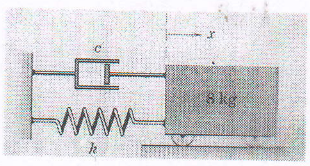
Notes

1- A critically damped system will have the smallest damping required for a periodic motion.

2- The mass returns to the position of rest in the shortest possible time without overshooting.

Example

The 8 kg body is moved 0.2 m to the right of the equilibrium position and released from rest at time $t = 0$. Determine its displacement at time



$t = 2$ sec, if the viscous damping constant c of 20 $\mathrm{N.sec/m}$, and stiffness k of spring is 32 $\mathrm{N/m}$.

Solution

Since the system is linear then, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{32}{8}} = 2\ rad/sec$

$\zeta = \frac{c}{2\omega_n m} = \frac{20}{2\cdot 2\cdot 8} = 0.625$ $\quad 0 < \zeta < 1$ underdamped

we need $\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.625)^2}\cdot 2 = 1.561\ rad/sec$ Note that $\omega_d < \omega_n$

$$x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

where $\phi = \tan^{-1}\frac{v_0 + x_0\zeta\omega_n}{x_0\omega_d}$ and $C = \sqrt{(x_0)^2 + \left(\frac{v_0 + x_0\zeta\omega_n}{\omega_d}\right)^2}$

$x_0 = +0.2$ m $\quad v_0 = 0$

$\phi = \tan^{-1}\frac{0+0.2\cdot 0.625\cdot 2}{0.2\cdot 1.561} = 38.68^\circ$ $\quad C = \sqrt{(0.2)^2 + \left(\frac{0+0.2\cdot 0.625\cdot 2}{1.561}\right)^2} = 0.256\ m$

so the general equation of the problem is

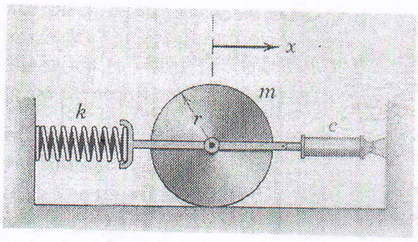
$$x(t) = 0.256e^{-1.25t}\cos(1.561t - 38.68^\circ)$$

then, when $t = 2\ sec$

$$x(t) = -0.0162\ m$$

Example

Derive the equation of motion for the homogeneous circular cylindrical which rolls without slipping. If the cylinder mass is 50 kg, the cylinder radius is 0.5m, the spring constant 75 $\mathrm{N/m}$ and the damping coefficient is 10 $\mathrm{N\cdot sec/m}$, determine :-



1- The undamped natural frequency (ω_n).

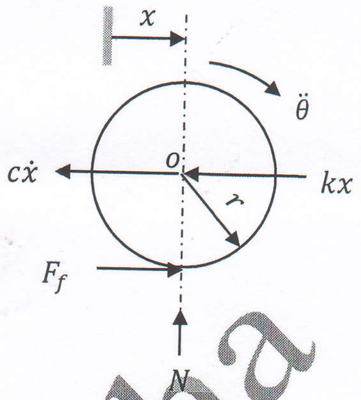
2- The damped ratio.

3- The damped natural frequency (ω_d).

4- The period of the damped system.

In addition, determine displacement x as a function of time if the cylinder is released from rest at the position $x = -0.2$m at time $t = 0$.

Solution



The disk has translation and rotation motions

Then from free body diagram

$$\sum F = m\,a_o = m\,\ddot{x}$$

$$-kx - c\dot{x} + F_f = m\,\ddot{x}$$

where F_f is friction force

$$m\,\ddot{x} + c\dot{x} + kx = F_f \qquad \cdots\cdots\cdots (a)$$

now, $\sum M_o = J_o\ddot{\theta}$ $\qquad \theta$:- is angular motion of the disk

$$-F_f r = J_o\ddot{\theta}$$

since $\quad x = r\theta \qquad \ddot{\theta} = \frac{\ddot{x}}{r} \qquad$ and $\quad J_o$ for disk $= \frac{1}{2}mr^2$

so $\quad -F_f r = \frac{1}{2}mr^2 \cdot \frac{\ddot{x}}{r} \qquad F_f = -\frac{1}{2}m\ddot{x}$

from Equation (a)

$$m\,\ddot{x} + c\dot{x} + kx = -\frac{1}{2}m\ddot{x}$$

$$\ddot{x} + \frac{2}{3}\frac{c}{m}\dot{x} + \frac{2}{3}\frac{k}{m}x = 0 \qquad \textbf{(Equation of Motion)}$$

comparing with the general form $\quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

$$\omega_n^2 = \frac{2}{3}\frac{k}{m} \qquad 2\zeta\omega_n = \frac{2}{3}\frac{c}{m} \qquad \zeta = \frac{1}{3}\frac{c}{m\omega_n}$$

$$\omega_n = \sqrt{\frac{2}{3}\frac{k}{m}} = \sqrt{\frac{2}{3}\cdot\frac{75}{50}} = 1\ rad/sec$$

If $\quad c = 10\ \mathrm{N\cdot sec/m}$

$$\zeta = \frac{1}{3}\frac{10}{50\cdot 1} = 0.0667 \quad \text{(under damped)}$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.0667)^2}\cdot 1 = 0.998\ rad/sec$$

$$T_d = \frac{2\pi}{\omega_d} = 6.3\ sec$$

$$x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

where $\quad \phi = \tan^{-1}\frac{v_0 + x_0\zeta\omega_n}{x_0\omega_d} \quad$ and $\quad C = \sqrt{(x_0)^2 + \left(\frac{v_0 + x_0\zeta\omega_n}{\omega_d}\right)^2}$

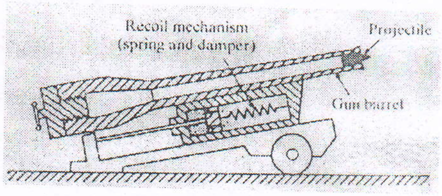
$$x_0 = -0.2\ \mathrm{m} \qquad v_0 = 0$$

$$\phi = \tan^{-1}\frac{0 - 0.2\cdot 0.0667\cdot 1}{-0.2\cdot 0.998} = 3.8236^\circ \qquad C = \sqrt{(-0.2)^2 + \left(\frac{0 - 0.2\cdot 0.0667\cdot 1}{0.998}\right)^2} = 0.2004\ m$$

$$x(t) = 0.2004\ e^{-0.0667t}\cos(0.998t - 3.8236^\circ)$$

Example

A sketch of a gun is shown in Figure. When the gun is fired, high-pressure gases accelerate the projectile inside the barrel to a very high velocity. The reaction force pushes the gun barrel in the opposite direction of the projectile. Since it is desirable to bring the gun barrel in the shortest time without oscillation, it is made to translate backwards against a critically damped spring – damper system called *recoil mechanism*. In particular case, the gun barrel and recoil mechanism have a mass of 500 kg with a recoil spring of stiffness 10000 N/m. The gun recoils 0.4 m upon firing. Find (a)- the critical damping coefficient of the damper, (b)- the initial recoil velocity of the gun, and (c)- the time taken by the gun to return to a position 0.1 m from its initial position.



Solution

Since the system is critically damped $\zeta = 1 = \frac{c_c}{2\omega_n m}$

$$c_c = 2\omega_n m$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{500}} = 4.4721\ rad/sec$$

so $c_c = 2 \cdot 500 \cdot 4.4721 = 4472.1\ \mathrm{N.sec/m}$,

for critically damping $x(t) = [x_0 + (\omega_n x_0 + v_0)t]\ e^{-\omega_n t}$

the initial conditions are $x_0 = 0$ $v_0 = ?$ but when $x = 0.4m$ $\dot{x} = 0$

$$x(t) = v_0 t\ e^{-\omega_n t}$$

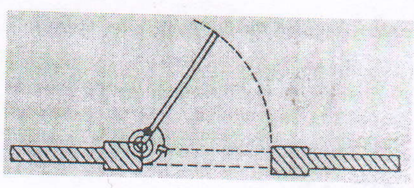
$$\dot{x}(t) = -\omega_n v_0 t\ e^{-\omega_n t} + v_0\ e^{-\omega_n t} \qquad 0 = -\omega_n v_0 t\ e^{-\omega_n t} + v_0\ e^{-\omega_n t}$$

$$\omega_n t = 1 \qquad t = \frac{1}{\omega_n} = \frac{1}{4.4721} = 0.2236\ \text{sec}$$

$$0.4 = v_0 \cdot 0.2236 e^{-4.4721 \cdot 0.2236} \qquad v_0 = 4.863$$

HW

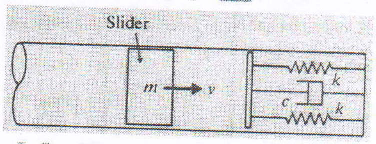
The door of a house has a height of $2m$, width of $1m$, thickness of $50mm$, and mass of 50 kg. the door opens against a torsion spring and viscous damper, as shown in Figure. If the spring constant of the torsional spring is 15 N.m/rad, find the damping



constant necessary to provide critical damping in the return swing of the door. If the door is initially opened 75° and released, how long will it take for the door to come to within 5° of closing

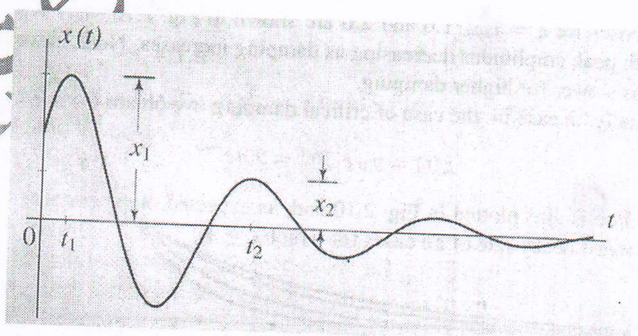
HW

A slider of mass 1 kg travels in a cylinder with a velocity $80\ \text{m/s}$ and engages a spring – damper system shown in Figure. If the stiffness of the spring is $k = 20\ \text{N/m}$ and the damping constant c of 0.02 N.sec/cm, find the maximum displacement of the slider after engaging the spring and damper. How much time does it take to reach the maximum displacement?



Measurement of Damping

A convenient way to determine the a mount of damping present in a system is to measure the rate of decay of free vibration.



x_1 :- is the amplitude at time t_1

x_2 :- is the amplitude at time t_2

where $t_2 = t_1 + T_d$

the general equation of oscillatory motion is

$$x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

when $t = t_1$ $x = x_1$

$$x_1 = Ce^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi)$$

and after one cycle $t = t_1 + T_d$

$$x_2 = Ce^{-\zeta\omega_n (t_1+T_d)}\cos(\omega_d (t_1 + T_d) - \phi)$$

$$\frac{x_1}{x_2} = \frac{Ce^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi)}{Ce^{-\zeta\omega_n (t_1+T_d)}\cos(\omega_d (t_1+T_d) - \phi)}$$ since $\cos(\omega_d t_1 - \phi) = \cos(\omega_d (t_1 + T_d) - \phi)$

$$\frac{x_1}{x_2} = e^{\zeta\omega_n T_d} \qquad \cdots\cdots\cdots (1)$$

$$\boxed{\ln\frac{x_1}{x_2} = \delta}$$ where δ is **logarithmic decrement**

so $\delta = \zeta\omega_n T_d = \zeta\omega_n \frac{2\pi}{\omega_d}$ but $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

$\delta = \zeta\omega_n \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$ $\delta^2 = \frac{(2\pi)^2\zeta^2}{1-\zeta^2}$

$$\boxed{\zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}}}$$

If ζ for the system is small then we can use the amplitudes x_1 and x_m [x_m is amplitude after m period], then

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\cdot\frac{x_2}{x_3}\cdot\frac{x_3}{x_4}\cdot\frac{x_4}{x_5}\,\ldots\ldots\ldots\,\frac{x_m}{x_{m+1}}$$

then from Equation (1) $\frac{x_i}{x_{i+1}} = e^{\zeta\omega_n T_d}$

$$\frac{x_1}{x_{m+1}} = \left(e^{\zeta\omega_n T_d}\right)^m = e^{m\zeta\omega_n T_d}$$

$$\ln\frac{x_1}{x_{m+1}} = m\,\underbrace{\zeta\omega_n T_d}_{\delta} = m\delta \qquad \boxed{\delta = \frac{1}{m}\ln\frac{x_1}{x_{m+1}}}$$ where m is integer

Example

An under damped shock absorber is to be designed for a vehicle. The initial amplitude is to be reduced to one – fourth in the first half cycle. The mass of the vehicle s 500 kg and the damping period of vibration is to be 1 sec. find the necessary stiffness and damping constants of the shock absorber. Also, if the clearance is 250 mm, find the minimum initial velocity that result in bottoming of the system.

Solution

$x_{1.5} = \frac{x_1}{4}$ $x_2 = \frac{x_{1.5}}{4}$ then $\frac{x_1}{x_2} = 16$

$\delta = \ln(16) = 2.7726$

$\zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}} = 0.4037$

$T_d = 1\,sec = \frac{2\pi}{\omega_d}$ $\Rightarrow$ $\omega_d = 2\pi = \sqrt{1-\zeta^2}\,\omega_n$

$2\pi = \sqrt{1-(0.4037)^2}\,\omega_n$ $\Rightarrow$ $\omega_n = 6.87\ rad/sec$

$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n}$ $c = 2\zeta m\omega_n = 2\cdot 500\cdot 0.4037\cdot 6.87 = 2772.5\ \mathrm{N.sec/m}$,

$$x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t-\phi) \qquad \ldots\ldots\ldots(a)$$

where $\phi = \tan^{-1}\frac{v_0+x_0\zeta\omega_n}{x_0\omega_d}$ and $C=\sqrt{(x_0)^2+\left(\frac{v_0+x_0\zeta\omega_n}{\omega_d}\right)^2}$

when, $t=0$, $x_0=0$ but $v_0\neq 0$

$$\phi=\tan^{-1}\infty \quad \Rightarrow \quad \phi=90^{\circ} \qquad C=\frac{v_0}{\omega_d}$$

$$\dot{x}(t)=-\zeta\omega_n Ce^{-\zeta\omega_n t}\cos(\omega_d t-\phi)-C\omega_d e^{-\zeta\omega_n t}\sin(\omega_d t-\phi)$$

$$0=-\zeta\omega_n Ce^{-\zeta\omega_n t}\cos(\omega_d t-\phi)-C\omega_d e^{-\zeta\omega_n t}\sin(\omega_d t-\phi)$$

$$\zeta\cos(\omega_d t-\phi)=-\sqrt{1-\zeta^2}\sin(\omega_d t-\phi)$$

$$\tan(\omega_d t-\phi)=-\frac{\zeta}{\sqrt{1-\zeta^2}}=-\frac{0.4037}{\sqrt{1-(0.4037)^2}}=-0.44125$$

$$\omega_d t-\phi=\tan^{-1}(-0.44125)$$

$$\omega_d t-\frac{\pi}{2}=-0.4155 \qquad \omega_d t=\frac{\pi}{2}-0.4155=1.1545$$

$$t=\frac{1.1545}{2\pi}=0.184 \text{ sec}$$

from Equation (a) when $t=0.184$ $x(t=0.184)=0.25$m

$$0.25=Ce^{-(0.4037\cdot 6.87\cdot 0.184)}\cos(2\pi.0.184-\frac{\pi}{2})$$

$$0.25=C\,(0.6)\,0.915 \qquad C=0.455$$

$$\boxed{x(t)=0.455e^{-2.77\,t}\cos\left(2\pi t-\frac{\pi}{2}\right)}$$

to find the initial velocity

$$\dot{x}(t)=-0.455\cdot 2.77e^{-2.77t}\cos\left(2\pi t-\frac{\pi}{2}\right)-0.455.2\pi e^{-2.77t}\sin\left(2\pi t-\frac{\pi}{2}\right)$$

now $\dot{x}(t=0)=0.455\cdot 2$

H.W1

A vibrating system consists of a mass of 4.534 kg, a spring of stiffness 35.0 N/cm, and a dashpot with a damping coefficient of 0.1243 $N.s/cm$. Find (a) the damping factor, (b) the logarithmic decrement, and (c) the ratio of any two consecutive amplitudes.

H.W2

A vibrating system consisting of a mass of 2.267 kg and a spring of stiffness 17.5 N/cm is viscously damped such that the ratio of any two consecutive amplitudes is 1.00 and 0.98. Determine (a) the natural frequency of the damped system, (b) the logarithmic decrement, (c) the damping factor, and (d) the damping coefficient.